**NUMERICAL METHODS   
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 6. Inverse problems for ordinary differential equations

We know that the inverse problems can be transformed to the problems of finding of extremum. So the practical methods of inverse problems theory are based on the optimization methods. The problems of the minimization of the functionals can be solved by means of the gradient methods. We know how we can use the gradient methods for linear abstract inverse problems. Now we consider this technique for the analysis of the inverse problems described by ordinary differential equations.

### 6.1. Differential equation with unknown absolute term

We consider the system be described by the equation

 (6.1)

with initial condition

 (6.2)

where  is the state function, and *y*0are known parameters, andis the unknown function. For all smooth enough function *v* the problem (6.1), (6.2) has a unique solution  We have also the equality

 (6.3)

where  is a given function (result of measuring). Determine the following inverse problem.

**Problem** **6.1**. *Find the function v such that the respective state function*  *satisfies the equality* (6.3)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **6.1'**. *Minimize the functional I.*

We know the relation between these problems. If *u* is a solution of Problem 6.1, then it satisfies the equality (6.3). So the respective value of the functional *I* is zero. Therefore this is the minimum of this functional because it does have any negative value. Thus the solution of the inverse problem is the solution of the optimization control problem. Then let *u* be a solution of Problem 6.1'. We can have two different cases. If the value  is zero, then the equality (6.3) is true; so the solution of the optimization control problem is the solution of the inverse problem. If the minimum of the functional is positive, then the inverse problem is insolvable, else there exist a parameter *v* with more small value than minimum. However the solution of Problem 6.1' can be chosen an approximate solution of Problem 6.1 because the equality (6.1) is realized in the best form. Hence we can transform the given inverse problem to the optimization control problem

We would like to use gradient method for solving Problem 6.2. The general step is the determination of the functional derivative. Determine the value



where *u* and *h* are functions, *σ* is a number. So we can find the difference

 (6.4)

Then we would like to determine of the difference  by means of the problem (6.1), (6.2).

Consider this system for the functions  and . We have

 (6.5)

 (6.6)

Multiply the equality (6.5) by an arbitrary function *λ* and integrate by *t*. We obtain

 (6.7)

After the integration by parts we get



because of the equality (6.6). So we obtain the equality

 (6.8)

Chose the function *λ* equal to the solution *p* of the differential equation

 (6.9)

with final condition

 (6.10)

where

 (6.11)

Therefore the relation (6.8) can be transformed to the equality



Put it to the equality (6.4). We get

 (6.12)

Prove that the second term at the right side of this equality has second order with respect to the parameter *σ*. Return to the equality (6.7). Chose  here. We get



Using the equality



we transform the previous equality. We get

 (6.13)

because of the equality  by (6.6).

We know the inequality



So we



Using this inequality, we transform the formula (6.13).



Then we have



We obtain



Devise this equality by *σ*. After the passing to the limit as  we get



Hence

 (6.14)

Devise the equality (6.12) by *σ*. We obtain



After the passing to the limit with using the equality (6.14) we have



Hence we find the derivative of the functional *I* at the point *u*

 (6.15)

We know the gradient methods for solving the problem of the minimization of a functional *I* on a unitary space

 (6.16)

where  is a positive iterative parameter. Using formula (6.15), we can determine the gradient method for our problem. Let *k-*iteration value *vk* of the control is known. Then we can find the appropriate value of the state function *yk* from Cauchy problem

 (6.17)

 (6.18)

Next step is the determination of the adjoint state *pk* from the adjoint equation (6.9), (6.10)

 (6.19)

 (6.20)

where

 (6.21)

The next iteration of the control can be determined by the formula (6.16) with using the formula (6.15) of the functional derivative. We get

 (6.22)

Hence we have the iterative algorithm (6.17) – (6.22) for solving our problem.

### 6.2. System of differential equations with unknown absolute term

We consider now the following system of differential equations

 (6.23)

with initial conditions

 (6.24)

where  is the state function, and are known parameters, is the unknown function, andis the unknown function. For all smooth enough function *v* the problem (6.23), (6.24) has a unique solution  We have also the equality

 (6.25)

where  is a given function (result of measuring). Determine the following inverse problem.

**Problem** **6.2**. *Find the function v such that the respective state function*  *satisfies the equality* (6.25)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **6.2'**. *Minimize the functional I.*

Find the derivative of this functional. Determine the value



where *u* and *h* are functions, *σ* is a number. Find the difference

 (6.26)

Determine of the difference  by means of the problem (6.23), (6.24). Consider this system for the functions  and . We have

 (6.27)

 (6.28)

where  Multiply the *i-*th equality (6.27) by an arbitrary function *λi*, add by *i*, and integrate by t. We obtain

 (6.29)

After the integration by parts we get



because of the equality (6.28). So we obtain the equality



Change the order of the addition; we get

 (6.30)

Chose the functions *λ*1, *λ*2 equal to the solution *p*1, *p*2 of the system of the differential equations

 (6.31)

with final condition

 (6.32)

Therefore the relation (6.30) can be transformed to the equality



Put it to the equality (6.26). We get

 (6.33)

This is the analogue of the equality (6.12).

Prove the smoothness of the second term at the right side of this formula. Return to the equality (6.7). Chose  here. We get



We have



because of the equality (6.28). So the previous formula can be transformed to the equality

 (6.34)

Suppose the matrix



is positive, that is



where *α* is a positive constant. Then the formula (6.34) can be transformed to the inequality



Using the inequality



we have



So we get



Then



namely



After devising the equality (6.33) by *σ* and passing to the limit we have



Hence we find the derivative of the functional *I* at the point *u*



We can apply the gradient method know. Let *k-*iteration value *vk* of the control is known. We determine the state function *y*1*k*, *y*2*k* from the system





Then we determine the solution of the adjoint system at the iteration *k*





The control at the next iteration is determined by the formula



### 6.3. Differential equation with unknown coefficient

We consider the system be described by the equation

 (6.35)

with initial condition

 (6.36)

where  is the state function, is a known smooth function, *y*0is a known parameters, and *v* is the unknown positive parameter. For all parameter *v* the problem (6.35), (6.36) has a unique solution  We have also the equality

 (6.37)

where  is a given function (result of measuring). Determine the following inverse problem.

**Problem** **6.3**. *Find the function v such that the respective state function*  *satisfies the equality* (6.37)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **6.3'**. *Minimize the functional I.*

We will solve this problem with using the gradient method. Find the derivative of the functional. Determine the value



where *u*, *h* and *σ* are numbers. So we can find the difference

 (6.38)

that is an analogue of the equality (6.4).

Consider this system for the functions  and . We have

 (6.39)

 (6.40)

where  The equality (6.39) can be transformed to



Multiply this equality by an arbitrary function *λ* and integrate by *t*. We obtain

 (6.41)

After the integration by parts we get



because of the equality (6.40). So we obtain

 (6.42)

Chose the function *λ* equal to the solution *p* of the adjoint system

 (6.43)

 (6.44)

where



Then the formula (6.42) can be transformed to the equality



Put it to the equality (6.38). We get

 (6.45)

Return to the equality (6.41). Chose  here. We get



Transform it to the equality



So we have



The number *u* and *h* are positive. So the term  will be positive for small enough *σ*. Then we obtain the inequality



By Schwartz inequality we have



Then we obtain



So we get



Then

 (6.46)

Devise the equality (6.45) by *σ*. We obtain

 (6.47)

We have the equality



because of the estimate (6.46). Besides we get



So



because of the estimate (6.46) too. Pass to the limit at the equality (6.47). We have



Then we find the derivative of the functional *I* at the point *u*



Determine the gradient method. Let the value *vk* is known. The state function *yk* is determine from Cauchy problem





Then we find the adjoint state from the system





where



The next iteration of the control is determined by the equality



### 6.4. Differential equation with two unknown parameters

We consider the system be described by the equation

 (6.48)

with initial condition

 (6.49)

where  is the state function, is a known smooth function. We do know the pair of numbers  besides *a* is positive. For all value *v* the problem (6.48), (6.49) has a unique solution  We have also the equality

 (6.50)

where  is a given function. Determine the following inverse problem.

**Problem** **6.4**. *Find the function v such that the respective state function*  *satisfies the equality* (6.50)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **6.4'**. *Minimize the functional I.*

Find the derivative of the functional. Determine the value



where  and *σ* is a number. So we can find the difference

 (6.51)

that is an analogue of the equalities (6.4) and (6.38).

Consider this system for the functions  and . We have

 (6.52)

 (6.53)

where  The equality (6.52) can be transformed to



Multiply this equality by an arbitrary function *λ* and integrate by *t*. We obtain

 (6.54)

After the integration by parts we get



because of the equality (6.53). So we obtain

 (6.55)

Chose the function *λ* equal to the solution *p* of the adjoint system





where



It is equal to the problem (6.43), (6.44). Then the formula (6.55) can be transformed to the equality



Put it to the equality (6.51). We get

 (6.56)

This is the analogue of the formula (6.45).

Return to the equality (6.54). Chose  here. We get



It can be transformed to



Then we obtain the inequality



that is the analogue of the formula (6.46). Therefore we can obtain again the conditions



Then we have



This is the scalar product  of the second order Euclid space between the vector  and the derivative . So this derivative is second order vector with components  and 

Determine now the gradient method. Let the parameters  are known. The state function *yk* is determine from Cauchy problem





Then we find the adjoint state from the system





where



The next iteration of the unknown parameters is determined by the equalities





### 6.5. Differential equation with pointwise measuring

We consider the system be described by the equation

 (6.57)

with initial condition

 (6.58)

where  is the state function, and *y*0are known parameters, andis the unknown function. For all smooth enough function *v* the problem (6.57), (6.58) has a unique solution  We have also the equalities

 (6.59)

where  are times of the experiments from the interval (0,*T*), and are a given values (results of measuring). Determine the following inverse problem.

**Problem** **6.5**. *Find the function v such that the respective state function*  *satisfies the equality* (6.59)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **6.5'**. *Minimize the functional I.*

Find the derivative of the functional. Determine the value



where *u* and *h* are functions, *σ* is a number. So we can find the difference



where .

Determine *δ*-function by the equality



Then we transform the previous equality to the formula



where *z* is an arbitrary function such that 

This *ϕ* difference satisfies the problem

 

This is the system (6.5), (6.6). So we determine the equality (6.7)



It can be transform to the equality (6.8)



Determine the adjoint system





Put  at the previous equality. We get

 (6.60)

We can prove the equality



After devising the equality (6.60) by *σ* and passing to the limit we have



So we find



Then we can determine the gradient method. Let the function *vk* is known. Then we find the function *yk* from Cauchy problem





Determine the adjoint state from the system





The next iteration of the control can be determined by the formula



### 6.6. Nonlinear differential equation with unknown absolute term

We consider the system be described by the equation

 (6.61)

with initial condition

 (6.62)

where  is the state function, *f* is known function of two arguments, *y*0are known parameters, andis the unknown function. Let this problem has a unique solution  We have also the equality

 (6.63)

where  is a given function (result of measuring). Determine the following inverse problem.

**Problem** **6.6**. *Find the function v such that the respective state function*  *satisfies the equality* (6.63)*.*

This problem can be transformed to the optimization control problem. Determine the functional



**Problem** **6.6'**. *Minimize the functional I.*

Find its derivative. Determine the value



where *u* and *h* are functions, *σ* is a number. So we can find the difference



where 

Consider the system with respect to the function *ϕ*

 (6.65)

 (6.66)

Let the function *f* be smooth enough. Than we can transform the term



where  *fy* is the partial derivative of the function *f* with respect to the first argument,  is the high term with respect to *ϕ*. Put it to the equality (6.65); we get



After multiplication by an arbitrary function *λ* and integration by *t* we obtain



After the integration by parts with using the initial condition (6.66) we get



Chose the function *λ* equal to the solution *p* of the system





where



Therefore the previous formula can be transformed to the equality



Put it to the equality (6.4). We get



It is possible to obtain the inequality



where the positive constant *c* does not depend from *σ*. Then we devise the previous equality by *σ* and pass to the limit as  We get



where  *fv* is the partial derivative of the function *f* with respect to the second argument. Hence we find the derivative of the functional *I* at the point *u*



We have the following iterative method. Let *k-*iteration value *vk* of the control is known. Then we can find the appropriate value of the state function *yk* from Cauchy problem





Next step is the determination of the adjoint state *pk* from the system





where



Then we find the new iteration of the control



Our next step is the analysis of inverse problems for mathematical physics problems.

### Task

Find the derivatives of the following functionals:



Determine the gradient method for these problems.